Matched Subspace Detectors for Stochastic Signals*

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Problem Statement

• The goal is to design detectors for stochastic signals or second-order signals.

• Extension of the first-order matched subspace detectors of Scharf and Friedlander.

• It is assumed that interference is nulled prior to processing by projecting the data into the space orthogonal to the interference subspace.

• We assume various states of knowledge about the parameters ?^2 and ?.
Pre-Processing

- In order to be invariant to the interference statistics, the data are projected into the space orthogonal to the interference.

- The data are then decomposed into their signal and noise components.

- The signal component is denoted by the vector $z$ and the noise component is denoted by the vector $w$.

\[
G = (I - P_S)H \\
z = (H^*(I - P_S)H)^{-1/2}H^*(I - P_S)y \\
= (G^*G)^{-1/2}G^*y
\]

\[
AA^* = I - P_S - P_G \\
w = A^*y
\]
Hypotheses

- The “noise” vector $w$ is distributed as a white complex Gaussian vector regardless of which hypothesis is in effect.

\[ f(w) = \frac{1}{(\pi\sigma^2)^{N-p}} e^{-\frac{w^*w}{\sigma^2}} \]

- Define $\Sigma = (G^*G)^{1/2}$. 

- When signal is present the data vector $z$ is distributed:

\[ f(z \mid \phi) = \frac{1}{(\pi\sigma^2)^p} e^{-\frac{1}{\sigma^2} \|z - \phi\|^2} \]

- When signal is not present the data vector $z$ is distributed:

\[ f(z \mid \phi = 0) = \frac{1}{(\pi\sigma^2)^p} e^{-\frac{z^*z}{\sigma^2}} \]
Likelihood Ratio

• For now, assume that the noise power \( \sigma^2 \) is known.

• In this case the vector \( \mathbf{w} \) is common to both hypotheses and is of no use.

• The conditional likelihood ratio is then

\[
l(\mathbf{z} \mid \phi; \sigma^2) = \frac{f(\mathbf{z} \mid \phi; \sigma^2)}{f(\mathbf{z} \mid \phi = 0; \sigma^2)} = \exp \left( \frac{\mathbf{z}^* \mathbf{z}}{\sigma^2} \right) \times \exp \left( -\frac{1}{\sigma^2} \| \mathbf{z} - \phi \|^2 \right)
\]
Unconditional Likelihood Ratio

- The unconditional likelihood ratio can be written as

\[ l(z; \sigma^2, \beta) = \exp \left( \frac{z^*z}{\sigma^2} \right) \times \int \exp \left( -\frac{||z-\phi||^2}{\sigma^2} \right) f_\phi(\phi; \beta) d\phi \]

- The log-likelihood ratio becomes

\[ s(z; \sigma^2, \beta) = \frac{z^*z}{\sigma^2} + \ln \int \exp \left( -\frac{||z-\phi||^2}{\sigma^2} \right) f_\phi(\phi; \beta) d\phi \]

\[ = \frac{y^*P_G y}{\sigma^2} - p_r(z; \sigma, \beta) \]

Matched Subspace Detector

Resolution penalty
Resolution Penalty

• The resolution penalty occurs because we presume to know something about the coordinate vector $\mathbf{z}$. 

• If $\mathbf{z}$ is far from the “favored” orientation defined by $\mathbf{z}$ then the penalty is larger than if the converse were true.

$$p_r(\mathbf{z}; \sigma^2; \beta) = -\ln \int \exp\left(-\frac{\|\mathbf{z} - \phi\|^2}{\sigma^2}\right) f_\phi(\phi; \beta) d\phi$$
Gaussian Coordinate Vectors

• Suppose \( \mathbf{\gamma} \sim \mathcal{CN}(0, \mathbf{R}) \).

• Write the eigenvalue decomposition of \( \mathbf{R} \) as:

\[
R_{\phi\phi} = (G^*G)^{1/2} R_{\theta\theta} (G^*G)^{1/2} = V D^2 V^*
\]

\[
V = [v_1 \ v_2 \ \cdots \ v_p]; \quad \text{unitary}
\]

\[
D^2 = \text{diag}[\beta_1^2, \beta_2^2, \cdots, \beta_p^2]
\]

• Define the resolved signal-plus-noise to noise ratios:

\[
r_i = 1 + \frac{\beta_i^2}{\sigma^2}
\]
Gaussian Penalty Term

• After some algebra the penalty term can now be written as

\[ p_r(z; \sigma^2, \beta^2) = -\ln \int \exp(-\frac{\|z-\phi\|^2}{\sigma^2}) \frac{1}{\pi^p \det(R_{\phi\phi})} \exp(-\phi^* R_{\phi\phi}^{-1} \phi) d\phi \]

\[ = \sum_{i=1}^{p} \ln(r_i) + \sum_{i=1}^{p} \frac{(z^* P_{v(i)} z / \sigma^2)}{r_i} \]

• This result implies that if the estimated signal-plus-noise to noise ratio \((z^* P_{v(i)} z / \sigma^2)\) in the resolved subspace defined by \(v_i\) greatly exceeds \(r_i\), then the penalty is large because of this mismatch.
Unknown Signal Power and Orientation

• Suppose that when signal is present we do not know $R$. 

• Recall the penalty term is

$$p_r(z; \sigma^2, \beta^2) = -\ln \int \exp(-\frac{\|z-\phi\|^2}{\sigma^2}) \frac{1}{\pi^d \det(R_{\phi\phi})} \exp(-\phi^* R_{\phi\phi}^{-1} \phi) d\phi$$

$$= \sum_{i=1}^{p} \ln(r_i) + \sum_{i=1}^{p} \frac{(z^* P_{V_i} z / \sigma^2)}{r_i}$$

• The estimates of the signal-plus-noise to noise ratios are

$$r_i = max(1, z^* P_{V_i} z / \sigma^2)$$

• We assume that $r_i \approx 1$ in the sequel.
Estimating Orientation

• The estimates of $r_i$ in the previous slide depend on the orientation of the vectors $v_i$.

• We want to minimize

$$\prod_{i=1}^{p} \frac{z^* P_{v_i} z}{\sigma^2}$$

• We must also satisfy the constraints

$$\sum_{i=1}^{p} \frac{z^* P_{v_i} z}{\sigma^2} = \frac{z^* z}{\sigma^2}$$

$$r_i = \frac{z^* P_{v_i} z}{\sigma^2} \geq 1$$
Intermediate Orientation Solution

• The solution to this optimization problem is

\[
    r_i = \frac{z^* P_{V_i} z}{\sigma^2} = 1 \quad \text{for} \quad i = 1, 2, \ldots, p - 1
\]

\[
    r_p = \frac{z^* P_{V_p} z}{\sigma^2} = \frac{z^* z}{\sigma^2} - (p - 1)
\]

• The question remains: Is there a decomposition of \( hG \) i that has the above properties?

• The answer is yes.
Orientation Solution

- Solve for \( v_p \) first.
- Choose a \( v_p \) on the spherical invariance set defined by

\[
\frac{z^* P_{v_p} z}{\sigma^2} = \frac{z^* z}{\sigma^2} - (p - 1).
\]

- Repeat this procedure in the spaces \( hA_{p-1} i, hA_{p-2} i, \ldots, hA_1 i \)

Great circle on invariance sphere

\[ \langle v_p \perp \rangle \]  
\[ \langle A_{p-1} \rangle \]  
\[ \text{Has norm } \sqrt{(p-1)} \]
Compressed Likelihood

- Compressing the likelihood ratio with this solution gives the statistic

\[ s(z; \sigma^2, \hat{R}_{\phi \phi}) = \frac{y^* P_H y}{\sigma^2} - \left[ \ln\left(\frac{y^* P_H y}{\sigma^2}\right) - \text{constants} \right] \]

- This statistic is a monotonic function of the matched subspace detector. We can therefore use the MSD as the detection statistic

\[ s = \frac{y^* P_H y}{\sigma^2} \]

- Then the result for 2\textsuperscript{nd}-order models is the same as for 1\textsuperscript{st}-order models.
Unknown Noise Power

- In the case of unknown noise power the GLRT detector can be written as a sum of the CFAR matched subspace detector and a penalty term

\[ s(z; \hat{\sigma}^2, \hat{R}_{\phi\phi}) = \ln(1 + \tilde{s}) - [\ln(\tilde{s}) - \text{constants}] \]

- We can equivalently use the statistic

\[ \tilde{s} = \frac{y^* P_H y}{\hat{\sigma}^2}; \quad \hat{\sigma}^2 = \frac{1}{N - p} y^* (I - P_H) y \]

- These detectors are identical to the 1\textsuperscript{st}-order results.
Rank-One Assumptions

• Here we assume that the signal subspace is rank-one.

• The complex-valued signal amplitude is written in polar form

\[ \theta = Me^{j\phi} \]

• Assume that the phase and magnitude are uncorrelated and that the phase is uniformly distributed over \([0, 2\pi)\).

• Assume that the signal magnitude has a generalized Rayleigh distribution

\[ f_M(M) = \frac{2M}{\beta^2} \left( \frac{M^2}{\beta^2} \right)^L \frac{e^{-M^2/\beta^2}}{L!} \]
Detectors with Known Noise

- **L=0.** This is the previous results with complex Gaussian amplitudes.

- **L ≠ 0.** The penalty function is

\[
p_r = (L + 1) \ln(r) + \frac{(y^* P_h y / \sigma^2)}{r} \left[ \ln \left( \sum_{k=0}^{L} \frac{\text{binom}(L,k)}{k!} \left( \frac{(y^* P_h y / \sigma^2)(r-1)}{r} \right)^k \right) \right]
\]

Minimize the penalty term with respect to \( r=1+\frac{?^2}{\sigma^2} \).

Compress the likelihood function with this term to obtain

\[
s = \frac{y^* P_h y}{\sigma^2} - p_r(\hat{r}).
\]