Space-Times Codes for an Invariant Detector of Frequency-Hopped MIMO Communications

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Codec Architecture for the Metachannel of an Invariant MIMO Detector

- **Multiple input multiple output (MIMO) communications**
  - Multiple transmitters coordinate channel coding by introducing space-time redundancy
  - Multiple receivers separate propagation modes in process of decoding
- **Frequency-hopped MIMO**
  - Channel transfer function (channel matrix) varies randomly hop-to-hop
  - Space-time coding occurs over hops and provides additional fading immunity and AJ
- **Invariant detector**
  - Short hops and low SNR can complicate channel estimation
  - Imposed detector invariances create metachannel robust to jamming and unknown channel

Walsh alphabet

Outer code: low density parity-check

Random channel: Hop-to-hop variations in channel matrix

Space-time inner code: matrix symbols

Parity-check matrix

Channel matrix eigenvalues

By quasi-maximum likelihood decoder

Parity-check matrix

Quasi-maximum likelihood decoder

Belief network

Walsh alphabet

\[
H = \begin{pmatrix}
1100 \\
0110 \\
0011
\end{pmatrix}
\]
Topics

- Introduction
- Signals in space
  - Signal model
  - Channel
  - Receiver
- Theoretical capacity
- Coding
  - Space-time inner codes
  - Low density parity-check outer codes
- Performance
  - Predictions
  - Simulations
- Summary and Conclusions
Subspace Codes

- Signal in additive noise (special case: # Rx = # Tx)

\[
\hat{Z} = V \hat{S} + N
\]

Assume \( l \geq 2n \)

- Motivation
  - In absence of noise, rowspace(\( Z \)) = rowspace(\( S \)) for nonsingular \( V \)
  - Encode information bits in subspaces rowspace(\( S \)) and use only subspace of observations
  - Decision invariant to whitening transformations \( Z \leftarrow R^{-1/2}Z \)

- Use scaled orthonormal signals (\( SS^H \propto I_n \)) to realize codes
Invariant Detectors

- Decision statistic $D(Z, S)$
- Invariances
  - Subspace invariance
    $$D(Z, S) = D(AZ, BS) \text{ for nonsingular } A, B$$
  - Independence, with Gaussian samples
    $$D(Z, S) = D(ZU, SU) \text{ for unitary } U$$
- Example:
  $$p(Z|R, V, S) = \pi^{-nl}|R|^{-l} \exp\{-\text{tr}[(Z - VS)^H R^{-1} (Z - VS)]\}$$
  $$p(AZ|R, V, BS) = |AA^H|^{-l}p(Z|A^{-1}RA^{-H}, AVB, T)$$
  $$p(ZU|R, V, SU) = p(Z|R, V, S)$$
  $$D(Z, S) \triangleq |ZZ^H|^l \cdot \max_{R,V} p(Z|R, V, S) \text{ has appropriate invariances}$$
- Maximal invariant $D(Z, S)$ depends only on principal angles between subspaces $\text{rowspace}(Z)$ and $\text{rowspace}(S)$
  - Other examples: $\text{tr}(P_Z P_S), |P_Z P_S|, \frac{|Z(I - P_S)Z^H|}{|ZZ^H|}$
Hopper Metachannel

- $V$ varies randomly hop to hop
  - Prior on $V$: mean zero, complex, unity variance Gaussian i.i.d. entries

- Channel model
  - Transmit rowspace ($S$)
  - Receive rowspace ($Z$), with $Z = aVS + N$

- Maximum likelihood detector ($p \triangleq |a|^2$)
  \[
  D(Z, S) = \left| I_n - \frac{p}{1 + p} P_Z P_S \right|^{-l}
  \]

- Channel capacity
  \[
  \mathbb{E}\left[ \log_2((1 + p)^{-l(l+1)/2} |I_n - \frac{p}{1+p} P_Z P_S|^{-l})/l \right]
  \]

- Suboptimal detector
  \[
  D(Z, S) = e^{\text{tr} P_Z P_S}
  \]
For \( m \) transmitters, \( n \) receivers, (average) data rate \( R \), average element-to-element SNR, and bandwidth \( B \), define \( \frac{E_b}{N_0} \) to satisfy

\[
mn \text{ SNR} = \frac{RE_b}{BN_0}
\]

Motivating properties

\[
\frac{E_b}{N_0} \to \log(2) \text{ as } B \uparrow \infty
\]

\[
\log 2 \leq \frac{E_b}{N_0} \text{ using average rate } R
\]

\[
m, n \to \infty, \frac{m}{n} \text{ fixed } \Rightarrow \log 2 = \frac{E_b}{N_0} \text{ for fixed rate } R
\]

Transmitted power proportional to \( \frac{1}{n} \frac{E_b}{N_0} \)

- MIMO \( \frac{E_b}{N_0} \) is \( n \) times MISO \( \frac{E_b}{N_0} \)
Capacity of the Metachannel

- Upper bound on capacity
  - Capacity when channel is tracked (known channel)
    \[ E_V [\log_2 (I_n + |a|^2 VV^\dagger)] \]

- Performance
  - As symbol length increases, capacity approaches that of tracked channel
  - Scaling all dimensions (number of receivers/transmitters and symbol length), channel behaves like infinite bandwidth channel but with added loss due to channel estimation.

Capacity of 4X4 MIMO As a Function of Symbol Length

![Graph showing capacity of 4X4 MIMO as a function of symbol length for different code lengths (8, 16, 32, 64) and channel types (Known Channel, 4 Tx-Rx Pairs).]
Space-Time Codes for the FH/PN Channel
Concatenated Coding

- Construct short space-time inner codes for each hop
  - Invariant to channel matrix
  - Matrix symbols with $2^m$ values
- Code over hops with low density parity-check (LDPC) outer code
  - Length 1024, rate $\frac{1}{2}$
  - 4 nonzero entries per column, 8 per row, totaling .8% of all entries
  - Symbols over GF($2^m$)
- Utilize invariant detector with probability vectors built from (quasi)-likelihoods

\[
I_n - \frac{p}{1+p} P_z P_s \left| \right|^{-1} e^{trlP_z P_s}
\]
Demodulating Matrix symbols

Mth hop matrix

Whiten

Kth symbol value

Correlators

\[ \hat{S}_{11} \]

\[ \hat{S}_{n1} \]

\[ \hat{S}_{1K} \]

\[ \hat{S}_{nK} \]

Vector Norms

\[ \| \cdot \|^2 \]

\[ \| \cdot \|^2 \]

\[ \| \cdot \|^2 \]

\[ \| \cdot \|^2 \]

Probability vector

\[ \Sigma \]

\[ \exp \]

\[ \Sigma \]

\[ \exp \]

LDPC decoder

\[ p_{m1} \]

\[ p_{mK} \]

Decision function:

\[ e^{trlP_Z P_S} \]
Decision Statistics For Matrix Symbols

- Quasi-likelihood and likelihood decision statistics provide similar performance
- Examples chosen from cases with about 5% symbol error probability
  - Histogram of components from length 16 probability vectors formed by (quasi)-likelihoods

Density of Suboptimal Quasi-likelihoods

\[ e^{trlP_Z P_S} \]

Density of Likelihoods

\[ \left| I_n - \frac{p}{1 + p} P_Z P_S \right|^{-l} \]
Graphical Decoding of Low Density Parity-Check Codes Using Bayesian Belief Networks

Variable dependencies

Loopless directed acyclic graph (DAG)
Directed Markov field
Bayesian belief network

$p(x_1, x_2, x_3, x_4) = p(x_4|x_3)p(x_3|x_1)p(x_2|x_1)p(x_1)$

Message passing protocol

Parent nodes (alphabets $u_k \in B$)
Update node (alphabet $x \in A$)
Child nodes

Node updates

Belief

$\pi(x) = \sum_u p(x|u_1, \ldots, u_z) \prod_{k=1}^{z} \pi_k^P(u_k)$

$\lambda(x) = \prod_k \lambda_k^C(x)$

$\pi_j^C(x) = \pi(x) \prod_{k \neq j} \lambda_k^C(x)$

$\lambda_j^P(u_j) = \sum_{x,u_k:k \neq j} \lambda(x)p(x|u_1, \ldots, u_z) \prod_{k \neq j} \pi_k^P(u_k)$

Network for a parity-check code

Bayesian Belief Network

Evidenciary nodes (observations)
Codeword components
Parity checks

Node firing order: $z\ 0\ c\ z\ 0\ c\ \ldots$
Stopping rule: parity check satisfied

Parity check matrix

$\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix}$
• Sets of orthonormal waveforms of length $l$: $\{C_k\}: C_j \perp C_k$

• Matrix symbols $S(c)$

$$\phi_k: GF(2^k) \rightarrow C_k, 1-1$$

$$c \in GF(2^k)^n$$

$$S(c) \triangleq \begin{pmatrix}
\phi_1(c_1) \\
\vdots \\
\phi_n(c_n)
\end{pmatrix}$$

• Spectral efficiencies ($r_s$, $r_t$ inner and outer code rates)

$$\frac{R}{B} = r_tr_s\frac{k}{2^k}$$
Examples of Space-Time Inner Codes

<table>
<thead>
<tr>
<th>Code</th>
<th>Parity Check Matrix</th>
<th>Field</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4,4,1)</td>
<td>0</td>
<td>$GF(2)$</td>
</tr>
<tr>
<td>(4,2,3)</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 1 &amp; 1 \ 0 &amp; 1 &amp; 1 &amp; \alpha \end{pmatrix}$</td>
<td>$GF(4)$</td>
</tr>
<tr>
<td>(4,3,2)</td>
<td>$(1,1,\ldots,1)$</td>
<td>$GF(2)$</td>
</tr>
<tr>
<td>(4,1,4)</td>
<td>$\begin{pmatrix} 1 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 1 \end{pmatrix}$</td>
<td>$GF(2)$</td>
</tr>
<tr>
<td>(8,8,1)</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 1 &amp; 1 &amp; \ldots &amp; 1 \ 0 &amp; 1 &amp; \alpha &amp; \alpha^2 &amp; \ldots &amp; \alpha^6 \end{pmatrix}$</td>
<td>$GF(8)$</td>
</tr>
<tr>
<td>(8,7,2)</td>
<td>$(1,1,\ldots,1)$</td>
<td>$GF(2)$</td>
</tr>
<tr>
<td>(8,6,3)</td>
<td>$\begin{pmatrix} 1 &amp; 1 &amp; 1 &amp; 1 &amp; 1 &amp; 1 &amp; 1 &amp; 1 \ 1 &amp; 1 &amp; 0 &amp; 1 &amp; 1 &amp; 0 &amp; 0 &amp; 0 \ 1 &amp; 1 &amp; 1 &amp; 0 &amp; 0 &amp; 1 &amp; 0 &amp; 0 \ 1 &amp; 0 &amp; 1 &amp; 1 &amp; 0 &amp; 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>$GF(2)$</td>
</tr>
<tr>
<td>(8,4,4)</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 1 &amp; 1 &amp; 1 \ 0 &amp; 0 &amp; \alpha &amp; \alpha^2 &amp; \alpha^3 \ 0 &amp; \ldots &amp; 0 &amp; \alpha^2 &amp; \alpha^4 &amp; \alpha^6 &amp; \alpha^9 \ 0 &amp; 0 &amp; \alpha^3 &amp; \alpha^6 &amp; \alpha^9 &amp; \ldots &amp; \alpha^{12} \end{pmatrix}$</td>
<td>$GF(8)$</td>
</tr>
<tr>
<td>(8,3,6)</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 1 &amp; 1 \ 0 &amp; 0 &amp; \alpha &amp; \alpha^2 \ 0 &amp; \ldots &amp; 0 &amp; \alpha^2 &amp; \alpha^4 \ 0 &amp; 0 &amp; \alpha^3 &amp; \alpha^4 &amp; \alpha^8 \ 0 &amp; 1 &amp; \alpha^4 &amp; \alpha^8 &amp; \alpha^{10} \end{pmatrix}$</td>
<td>$GF(8)$</td>
</tr>
<tr>
<td>(8,2,7)</td>
<td>$\begin{pmatrix} 1 &amp; 1 &amp; 0 &amp; 0 &amp; \ldots &amp; 0 \ 0 &amp; 1 &amp; 1 &amp; 0 &amp; \ldots &amp; 0 \ \vdots \end{pmatrix}$</td>
<td>$GF(2)$</td>
</tr>
<tr>
<td>(8,1,8)</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; \ldots &amp; 1 &amp; 1 \end{pmatrix}$</td>
<td>$GF(2)$</td>
</tr>
</tbody>
</table>
More of Space-Time Inner Codes
Steiner Systems

- Orthonormal waveforms
  \[ \{ \tilde{s}_k \} , \; \tilde{s}_j \perp \tilde{s}_k , \; j \neq k , \; 1 \leq j, k \leq l \]

- Matrix symbols
  \[ E(c) \triangleq \begin{pmatrix} \tilde{s}_{i_1} \\ \vdots \\ \tilde{s}_{i_n} \end{pmatrix} \]

  \{ c_{i_1}, \ldots , c_{i_n} \} \; \text{nonzero entries in } C, \; \text{wt}(c) = n

- Examples
  \[ C = \left\{ \begin{array}{l}
  (l = 16, 11, n = 4) \; 140 \; \text{codewords} \\
  (l = 24, 12, n = 8) \; 759 \; \text{codewords}
  \end{array} \right\} \; \text{wt}(c) = n \]

- Subspace separations
  \[ \dim(E(c) \cap E(c')) \leq \left\{ \begin{array}{l}
  2 , \; (16, 11, 4) \\
  4 , \; (24, 12, 8)
  \end{array} \right\} \; c \neq c' \]

  Maximally separated away from intersection
Theoretical Predictions
Approximate Error Exponents

- Effective SNR (interference covariance $R_I$ as r.v. hop to hop)
  \[
  \frac{nd}{4} \left( \frac{\text{tr}(E[R_I^{-1}VV^H])}{n^2} \right)^2 \cdot (\text{ISNR})^2
  \]

- Bounds for linear block codes ($D/N \leq 1/2$)
  Gilbert-Varshamov (GS):
  \[
  \sum_{k=0}^{D-2} (q - 1)^k \binom{N - 1}{k} < q^r
  \]
  Rank: $D \leq N - K + 1$

- Asymptotic form of Gilbert-Varshamov bound
  \[
  G_q(x) \triangleq \log q - x \log(q - 1) - x \log x - (1 - x) \log(1 - x)
  \]
  \[
  \frac{K}{N} \log q = G_q(D/N)
  \]

- Error exponent (GS):
  \[
  \frac{K}{N} \log q - \text{SNR}_{\text{eff}} G_q^{-1} \left( \frac{K}{N} \log q \right)
  \]
Comparison of Theoretical and Simulated Performance

- Predicted performance expresses code rate in terms of SNR
- Minimizing $\frac{E_b}{N_0}$ over SNR results in optimal codes of rate near 1/2
- Predicted performance agrees closely with simulated 1/2 rate LDPC outer code concatenated with space-time inner codes

**Block code bounds**

- Code and field (4,2,3) GF(4)
- Minimal $E_b/N_0$ and rate 10 dB 0.4
- Spectral efficiency 0.1 bits/Hz/sec

**Code rate (k/n)**

Predicted performance
Simulated 1/2 rate LDPC

**Element SNR (dB)**

Predicted performance
Simulated 1/2 rate LDPC

**Block code bounds**

- Code and field (8,4,4) GF(2)
- Minimal $E_b/N_0$ and rate 14 dB 0.42
- Spectral efficiency 0.1 bits/Hz/sec

**Element SNR (dB)**

Predicted performance
Simulated 1/2 rate LDPC

**4x4 ↔ MIMO ↔ 8x8**

Predicted performance
Simulated 1/2 rate LDPC
Simulated Performance With Jamming and Nonrandom Channel Matrices

- Theoretically, K jammers result in \( (N-K)/N \) SINR loss
- Simulated results indicate losses are somewhat higher
- When channel matrix is constant over all hops, predicted performance agrees with random variation provided received power is scaled to make \( \text{tr}(VV^\dagger)/n^2 \) unity

Simulated performance with and without jamming

\( 4\times4 \) \( \xrightarrow{\text{MIMO}} \) \( 8\times8 \)

- Code: (4,2,3)
- White noise
- 2 cochannel Interferers
- Theoretical loss in interference: 3 dB

- Code: (8,4,4)
- White noise
- 4 cochannel interferers
- Theoretically, 5 dB
Summary of Performance
Random Channel Matrices

- **Codes**
  - Inner code specified by block code parameters, Steiner system parameters or random matrix symbols (4X4 MIMO with 16 length 16 matrix symbols)
  - Outer code: (1024,512) LDPC over GF(16), GF(128), or GF(256)

- **Performance**
  - Predicted by effective SNR and Gilbert-Varshamov bounds (except random case)
  - Bounds validated by simulation (within several tenths dB)

```
Eb/N0 (dB)  
0 2 4 6 8 10 12 14

Code capacity

Shannon capacity

Codec capacity

Spectral efficiency (bits/Hz/sec)
```

(1,1,1) Block  1X1
(4,1,4) Block  4X4
(4,4,1) Block  4X4
(4,2,3) Block  4X4
(8,4,4) Block  8X8
(16,11,4) Steiner  4X4
random  4X4
Summary and Conclusions

- Class of invariant detectors formulated for robust demodulation and decoding in unknown interference with unknown channels
  - Capacity evaluated for the frequency-hopped (FH) channel as received by an invariant detector

- Family of concatenated codes examined for frequency-hopped, pseudo-noise (FH/PN) channel
  - Family uses linear block codes, Steiner systems, etc. for space-time inner code matrix symbols and low density parity-check outer codes
  - Theoretical performance agrees with simulations

- Performance
  - Concatenated codes considered operate around 3 to 4 dB (MISO) $\frac{E_b}{N_0}$
  - Concatenated codes examined are 7-8 dB worse than channel capacity bound in white noise
  - Space-time codes provide $n^2$ diversity even when channel matrices remain constant hop to hop
  - Space-time codes and invariant detector handle interferers and unknown channels gracefully with little sensitivity to interference geometry