ABSTRACT

To achieve high signal-to-noise ratios (SNR) while maintaining moderate sensor size, an architecture is proposed to combine several independent radar apertures into a coherently functioning unit. The proposed system utilizes several distinct apertures which are to be adaptively cohered based on observed target returns. Two operating modes are presented (in addition to each aperture operating as an independent radar). In the first, a Multiple Input Multiple Output (MIMO) mode, each aperture transmits a distinct waveform. The independent, multi-static returns allow the estimation of several coherence parameters. These estimates may then be used to produce a fully coherent transmit/receive mode, wherein all of the apertures cooperatively transmit a waveform which allows the system to perform as a fully coherent, sparse array.

The success of each of these modes, particularly the coherent transmit/receive mode, is dependant upon accurate estimates of several coherence parameters. In particular, the one-way travel times between each aperture and the target and the local oscillator phases of each aperture must be precisely estimated from observed returns. This paper examines the estimation problem in the context of the Cramer-Rao lower bound. Based upon this theoretical bound on the estimate accuracy, the desired performance gain lies within the theoretically achievable realm using the adaptive coherence process.

1. INTRODUCTION

Long-range search, tracking, and discrimination require radar returns with high signal-to-noise ratios (SNR) and large bandwidths. High SNR requirements lead to high power radars with large apertures. Large apertures and high bandwidth waveforms necessitate advanced processing. Specifically, the high time-bandwidth product associated with a large wide-band array leads to a significant performance loss when processing with basic phased array steering. This performance loss is well understood, and is generally mitigated with time delay steering. Time delays applied at each element properly steer a wide-band beam, but hardware required is complicated and costly. Commonly, the array is divided into sub-arrays with time delay compensation applied on the sub-array level. The sub-array architecture seeks to minimize steering loss while maintaining acceptable hardware simplicity.

The large size of search and track radars has a secondary impact on their practical value, as they are quite difficult to transport. Ideally, such radars would be readily deployable. The sub-array architecture described above offers an interesting potential to solve the transportability challenge while maintaining the desired performance. Consider several transportable apertures, each functioning as an independent radar. By utilizing wide-band communication channels between the apertures, the independent radars may be treated as sub-arrays of a single large-aperture radar. The sub-apertures may be abutted, creating a continuous combined aperture, or they may be separated, leading to a larger, though sparse, array. We will refer to the architecture as a sparse, or distributed, aperture where abutted sub-apertures form a simple case.

The most immediate challenge of the distributed architecture is to coherently combine the returns from each sub-aperture. Coherence in a large array is traditionally achieved by accurate knowledge of relative element locations. With precise a priori knowledge of element position, one may calculate the effect of an incident wave. For the distributed architecture, element positions are not relatively fixed, complicating calibration. While array locations may be measured, location and orientation measurements must be extremely precise to provide a priori coherence parameters. Furthermore, the sub-arrays are each driven by an independent, though stable, clock and local oscillator (LO), which creates significant timing issues.

In this paper we propose a target-based calibration method. The distributed aperture can operate in several modes, with the simple modes generating the necessary
information for more advanced modes. For example, each sub-aperture can function independently with little communication between the sub-apertures. Although this provides operational flexibility, it is not discussed further in this paper. The mode of interest here is a cooperative mode, where each sub-aperture transmits a distinct member of a set of orthogonal waveforms. Each sub-aperture then receives all of the transmitted waveforms. This may be thought of as several multi-static radars, or a multiple-input multiple-output (MIMO) radar. The MIMO mode may estimate the necessary calibration parameters to steer a coherent beam towards the target of interest, thus providing the necessary input for a fully coherent mode.

This paper is divided into two main sections. In the first, we will derive the signal model for both the MIMO mode and the coherent transmit/receive mode. This section will discuss the parameters needed for on-target coherence processing. In the second section, we will apply the Cramer-Rao performance bound to each of the modes, to examine the theoretical limits on performance.

2. COHERENCE PROCESSING

For the distributed aperture, multiple channels are generally mutually non-coherent. Each channel represents a target return that may differ in several ways. First, each channel represents a distinct physical path, which means that both the distances traveled and atmospheric effects may be different. Secondly, distinct channels may display different scattering characteristics from the target. Finally, different channels represent returns from distinct transmitters and receivers, and thus from different local oscillators. This introduces a distinct insertion phase into each channel’s response.

We will model the mutual incoherence of the channels with two parameters. Each channel is affected by a time delay and a phase offset. This can be viewed as modeling the different path-lengths and different local oscillators. We will assume that each of the apertures may be considered sufficiently collocated to neglect differences in target scattering and atmospheric effects.

2.1 MIMO mode (Coherent Receive)

If there are \( N \) apertures to be used cooperatively, each one simultaneously transmits an orthogonal waveform. In concept, these waveforms would have the same center frequency and the same bandwidth, but have orthogonal responses. If \( s_i(t) \) represents the \( i \)-th waveform (\( i = 1, \ldots, N \)), we model the orthogonal nature of the waveforms by

\[
s_j(t) * s^*_i(-t) = \beta(t) \delta_{ij}(t),
\]

where \( * \) represents the convolution operator, \( \beta(t) \) represents the main-lobe response (assumed to be the same for all \( N \) waveforms), and

![Figure 1. Block Diagram for MIMO operation mode. Orthogonal waveforms allow each transmit/receive channel to be isolated at the central processor. This allows sub-array steering and coherence parameters to be applied in post processing.](image)

\[
\delta_{ij}(t) = \begin{cases} \delta(t), & i = j \\ 0, & i \neq j \end{cases}
\]

In actuality, the waveforms are not orthogonal. We may model this by introducing a term \( \gamma(t) \) that represents the cross-channel leakage as well as the side-lobe response. Each aperture transmits one of the set of orthogonal waveforms, and each aperture receives all \( N \) waveforms. This is shown in block diagram in figure 1. In this way, there are \( N^2 \) channels of target returns. If we assume that each channel observes the same SNR, coherent combination of the channels may potentially generate an SNR gain of \( N^2 \).

In order to evaluate the performance of this mode, we consider the signal at each step in the block diagram. For ease in analysis, all signals will be represented as continuous time signals, though implementation in a real system will certainly contain A/D and D/A converters.

The \( j \)-th transmitter generates the waveform \( s_j(t) \). This is mixed up with local oscillator \( e^{-j\omega_o t + j\theta^j} \), producing, as a result \( s_j(t)e^{-j\omega_o t + j\theta^j} \). Note that \( \omega_o \) represents the carrier frequency and \( \theta^j \) represents the phase (on transmit) of the \( j \)-th local oscillator.

The waveform propagates through free-space to the \( i \)-th receiver over a time delay \( \tau_i = \tau_r + \tau_f \) (where \( \tau_r \) represents the one-way time delay from the target to the \( i \)-th aperture) and is scattered by a complex scatterer with response \( a_{ij} \), producing

\[
a_{ij}s_j(t - \tau_f)e^{-j\omega_o(t - \tau_f) + j\theta^j}.
\]

The \( i \)-th receiver, in free space, observes the superposition of all \( N \) orthogonal waveforms:

\[
A_i(t) = \sum_{k=1}^{N} a_{ik}s_k(t - \tau_k)e^{-j\omega_o(t - \tau_k) + j\theta^k}.
\]
The matched channel is mixed down with local oscillator $e^{j\omega t + j\beta t}$:

$$A(t) = \sum_{k=1}^{N^2} a_k s_k(t - \tau_k) e^{-j\omega t + j\beta t}.$$  

At this point, the orthogonal property of the waveforms are exploited as each receiver may match to a specified transmit waveform. We will also employ a simplifying assumption about the target and return paths, namely that the target is non-scintillating and we may assume $a_i = a_j$ for all $i,j$. Matching to the $j$th receive channel,

$$s_j^*(-t) A(t) = a \beta(t - \tau_y) e^{j\omega(t - \tau_y) + j\beta t} + \sum_{k=1}^{N^2} a_k s_k^*(t - \tau_k) e^{j\omega(t - \tau_y) + j\beta t} + n_j(t).$$

Though we will return to them later, for now we will disregard the side-lobe and cross-channel leakage terms. From all of the receive channels, estimates are generated for $\hat{\tau}_y = \hat{\tau}_y + \hat{\phi}_y$, and $\hat{\psi}_y = \hat{\omega}_y \hat{\tau}_y + \hat{\theta}_y$. The received pulses are time aligned and multiplied by the phase term so that the channels will add coherently:

$$A_j(t + \hat{\tau}_y) e^{-j\hat{\psi}_y} = a \beta(t - \hat{\tau}_y) e^{j\hat{\psi}_y} + n_j(t).$$

(Note that the time and frequency shifts, as well as the phase multipliers have been dropped, without loss of generality, from the noise terms.) By applying a variable transformation of $\varepsilon_y = \tau_y - \hat{\tau}_y$ and $\phi_y = \psi_y - \hat{\psi}_y$, we may simplify this expression to

$$A_j(t + \hat{\tau}_y) e^{-j\varepsilon_y} = a \beta(t - \varepsilon_y) e^{j\phi_y} + n_j(t).$$

The coherently combined response of the $N^2$ channels is then

$$A(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} A_j(t + \hat{\tau}_y) e^{-j\varepsilon_y} = a \sum_{i=1}^{N} \sum_{j=1}^{N} \beta(t - \varepsilon_y) e^{j\phi_y} + n_j(t).$$

### 2.2 Coherent Transmit/Receive Mode

The MIMO mode may be considered as a means of bootstrapping to obtain greater coherent gain. By transmitting the same waveform coherently from all $N$ apertures, we may potentially achieve a gain approaching $N^2$. If time delays and phases are stable, then by utilizing the estimates for $\hat{\psi}$ and $\hat{\tau}$ obtained via the MIMO mode, we may apply phase shifts and time delays on transmit. (Note that applying time delays of $\hat{\tau}$ is the equivalent of the standard time-delay steering.) This operation is shown in block diagram in figure 2.

![Figure 2. Block diagram for coherent transmit/receive operation mode. Steering and coherence parameters observed in the MIMO mode are applied on transmit and receive paths for each sub-array. This mode emulates a fully coherent aperture with corresponding performance.](image)

The $f$th transmitter generates the waveform $s(t)$. (Note that there is no subscript, implying that the same waveform is generated by each transmitter.) We apply a time delay $\hat{\tau}_f$ and phase $\hat{\psi}_f = \hat{\theta}_f + \omega \hat{\tau}_f$ based on previously obtained estimates:

$$s(t + \hat{\tau}_f) e^{-j\hat{\psi}_f} e^{-j\omega \hat{\tau}_f}.$$  

This is mixed up with local oscillator $e^{-j\omega t}$

$$s(t + \hat{\tau}_f) e^{-j\psi} e^{-j\omega t}.$$  

The waveform propagates in free to the $i$th receiver, thus undergoing a time delay of $\tau_i = \tau_f + \hat{\tau}_f$. The $i$th receiver observes in free space the combination of all $N$ waveforms transmitted from the separate apertures:

$$A_i(t) = \sum_{f=1}^{N} s(t + \hat{\tau}_f) e^{-j\psi} e^{-j\omega t}.$$  

The waveform is mixed down with local oscillator $e^{j\omega t + j\beta t}$ and matched filtered

$$A_i(t) = \sum_{f=1}^{N} \left( a \beta(t + \hat{\tau}_f) e^{j\omega t + j\beta t} + n_f(t) \right).$$

We apply a time delay $\hat{\tau}_i$ and phase $\hat{\psi}_i$ to the received signals based on previously obtained estimates:

$$A_i(t + \hat{\tau}_i) e^{-j\hat{\psi}_i} = \sum_{f=1}^{N} \left( a \beta(t + \hat{\tau}_f) e^{j\omega(t + \hat{\tau}_f) + j\beta t + j\hat{\theta}_f} e^{j\hat{\omega} t} \right) + n_f(t).$$

Applying the same variable transformation as above, this simplifies to

$$A_i(t + \hat{\tau}_i) e^{-j\hat{\psi}_i} = \sum_{f=1}^{N} \left( a \beta(t - \varepsilon_f) e^{j\phi_f} \right) + n_f(t).$$
Coherent combination of these \( N \) channels is the summation of each of the receiver outputs:

\[
A(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} a \beta(t - \varepsilon_j) e^{j\phi_i} + \sum_{i=1}^{N} n_i(t).
\]

### 2.3 SNR Calculations

We have derived a combined channel response for both the MIMO mode and the Coherent T/R mode. Given these results, we may generate an expression for the expected SNR for each mode. One may see, in fact, that the two expressions are nearly identical – the signal terms are identical (\( \sum_{i=1}^{N} \sum_{j=1}^{N} a \beta(t - \varepsilon_j) e^{j\phi_i} \)) while the difference is the number of noise channels to be summed. The noise power in the MIMO and the Coherent T/R modes are, respectively \( N^2 \sigma^2 \) and \( N \sigma^2 \). For both cases, to calculate the expected power in the signal, we must evaluate

\[
E \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} |a| \beta(\varepsilon_i) \beta(\varepsilon_j) e^{j(\phi_i - \phi_j)} \right].
\]

If we assume independence between \( \varepsilon \) and \( \phi \), then the summation becomes

\[
|a|^2 \sum_{i=1}^{N} \sum_{j=1}^{N} E \left[ \beta(\varepsilon_i) \beta(\varepsilon_j) \right] E \left[ e^{j(\phi_i - \phi_j)} \right] .
\]

The two expectations represent the losses due to range misalignment and the losses due to phase mismatch. For perfect range alignment and phase match, the expected signal power would be \( N^4 |a|^4 \), thus the SNR for the MIMO and Coherent T/R modes would be \( N^2 |a|^2 / \sigma^2 \) and \( N^4 |a|^4 / N \sigma^2 \). Appendix A contains details of the expectation.

The impact of the cross-channel leakage term for the MIMO mode, \( \gamma(r)(t) \), should be noted here. If we model this term as a noise-like response with variance \( \sigma^2 \), then the SNR for the MIMO mode becomes \( N^2 |a|^2 / (\sigma^2 + \sigma^2) \). In practice, the cross-channel leakage terms will only impact the performance when the noise floor drops below the expected side-lobe level of the waveforms, i.e. when \( \sigma^2 \ll \sigma^2 \) or \( \sigma^2 \approx \sigma^2 \).

### 3. PERFORMANCE BOUNDS

Given the above expression for SNR and SNR gain for the various modes, what remains is to include realistic values for the variance of the random variables \( \varepsilon \) and \( \phi \). (We have assumed that these variables are normally distributed.) An expression for the Cramer-Rao lower bound for a single wideband radar has been derived [1]. This result can be extended to the MIMO estimates, and thus provide an upper limit to the SNR gain.

Consider a single channel in the MIMO mode. The Cramer-Rao bound provides a lower bound (as a function of SNR) on the variance of the estimates \( \hat{\tau}_i \) and \( \hat{\psi}_i \). In the MIMO mode, there are \( N^2 \) channels, and thus \( N^2 \) observations \( \hat{\tau}_i \) and \( \hat{\psi}_i \). However, as derived above, there are only \( N \) underlying parameters \( \tau_i \) and \( \psi_i \) for the un-calibrated case: \( \tau_i, \tau_i', \psi_i, \) and \( \psi_i' \). It stands to reason, then, that the estimates for these underlying parameters improves as \( N \) increases, as the number of observations increases by \( N^2 \), but the number of parameters as \( N \) or \( 2N \). It can be shown by straightforward linear algebra that the MIMO estimates of these underlying parameters have a variance decreased by a factor of \( (2N-1)/4N^2 \) for the calibrated case and \( (4N-3)/4N^2 \) for the un-calibrated case.

Given this expression for the lower bound on range alignment and phase match, we may generate an upper bound on SNR gain, as a function of single sensor SNR. We have derived an expression for the combined channel SNR, requiring input of variance on \( \varepsilon \) and \( \phi \), as well as selecting a main-beam pulse shape, \( \beta \). While we may use any model for the pulse shape, two reasonable choices are
a sinc waveform and a triangle waveform. The sinc represents the traditional class of waveforms, as an expected response, while the triangle is a common model for pseudo-noise waveforms used to approximate orthogonality.

The results are shown for $N=2,3$, and 5 for both modes in figure 3.

4. CONCLUSION

Coherent combining of sparse apertures is an appealing alternative to generating a single large aperture. Several smaller apertures offer flexibility and transportability. By having a Multiple Input Multiple Output (MIMO) mode, the sparse aperture has the potential to achieve significant SNR gains as well as providing coherence parameters for fully coherent transmit and receive. The necessary SNR to achieve such target-based coherence is within the region of interest, as calculated using the Cramer-Rao bound. If the SNR is high enough, coherent combining is not needed. If the SNR is too low, we do not perform initial detection. The gain detailed through the coherent combining allows reduction in integration time and higher SNR after the initial target acquisition. The theoretical bounds and the performance enhancements warrant further effort to demonstrate the realizability of the architecture.

5. REFERENCES


Acknowledgement

The authors would like to thank J. Russell Johnson at MIT Lincoln Laboratory for guidance in the derivation of this model and the performance bounds.

APPENDIX A

To evaluate the expected SNR gain requires the evaluation of an expectation of the coherent sum of $N^2$ channels as given by the following equation:

$$\mathbb{E}\left[\sum_{i}^{N} \sum_{j}^{N} \sum_{i'}^{N} \sum_{j'}^{N} E\{\beta(\varepsilon_i)\beta(\varepsilon_j)\} E\{e^{i\phi_{ij}}\}\right]$$

Understanding the loss various terms requires expressions for the expectations for various values of $i, j, i', j'$. Combinations of these indices affect the distributions of the random variables $\varepsilon$ and $\phi$, specifically affecting the variance and covariance. We may think of the problem as summing all four-tuples drawn from $N$ samples. To evaluate the total we must consider each of the distinct cases of the four-tuple $\{i \ j \ i' \ j'\}$, drawn with replacement from $N$ samples. There are $N^4$ total realizations, but these fall into several statistically distinct cases. Each case is considered in table form, both for the calibrated and the un-calibrated cases.

<table>
<thead>
<tr>
<th>Description</th>
<th>Number of occurrences</th>
<th>$E\beta(\varepsilon_i)\beta(\varepsilon_j)$</th>
<th>$Ee^{i\phi_{ij}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>All four paths the same</td>
<td>$N$</td>
<td>$E\beta(\varepsilon_i)\beta(\varepsilon_j)$, $\varepsilon \sim N(0,4\sigma^2)$</td>
<td>1</td>
</tr>
<tr>
<td>Three of the four paths are the same</td>
<td>$4N(N-1)$</td>
<td>$E\beta(\varepsilon_i)\beta(\varepsilon_j)$, $\varepsilon \sim N(0,4\sigma^2)$, $E\varepsilon E_{\varepsilon} = 2\sigma^2$</td>
<td>$e^{-c}$</td>
</tr>
<tr>
<td>Two of the four paths are the same: $i = j \ xor \ i' = j'$</td>
<td>$2N(N-1)(N-2)$</td>
<td>$E\beta(\varepsilon_i)\beta(\varepsilon_j)$, $\varepsilon \sim N(0,4\sigma^2)$, $E\varepsilon E_{\varepsilon} = 0$</td>
<td>$e^{-c}$</td>
</tr>
<tr>
<td>Two of the four paths are the same: $i = i' \ xor \ j = j'$</td>
<td>$4N(N-1)(N-2)$</td>
<td>$E\beta(\varepsilon_i)\beta(\varepsilon_j)$, $\varepsilon \sim N(0,4\sigma^2)$, $E\varepsilon E_{\varepsilon} = \sigma^2$</td>
<td>$e^{-c}$</td>
</tr>
<tr>
<td>Two of the four paths are the same: $i = j$ and $i' = j'(i \ neq i')$</td>
<td>$3N(N-1)/2$</td>
<td>$E\beta(\varepsilon_i)\beta(\varepsilon_j)$, $\varepsilon \sim N(0,4\sigma^2)$, $E\varepsilon E_{\varepsilon} = 0$</td>
<td>$e^{-c}$</td>
</tr>
<tr>
<td>Two of the four paths are the same: $i = i'$ and $j = j'(i \ neq j)$</td>
<td>$3N(N-1)/2$</td>
<td>$E\beta(\varepsilon_i)\beta(\varepsilon_j)$, $\varepsilon \sim N(0,4\sigma^2)$</td>
<td>1</td>
</tr>
<tr>
<td>All four paths are distinct</td>
<td>$N(N-1)(N-2)(N-3)$</td>
<td>$E\beta(\varepsilon_i)\beta(\varepsilon_j)$, $\varepsilon \sim N(0,4\sigma^2)$, $E\varepsilon E_{\varepsilon} = 0$</td>
<td>$e^{-c}$</td>
</tr>
</tbody>
</table>

Table 1. Expectations for calibrated cases.
<table>
<thead>
<tr>
<th>Description</th>
<th>Number of occurrences</th>
<th>$\beta(\varepsilon) \beta(\varepsilon')$, $\varepsilon - N(0, 2\sigma')$, $\varepsilon - N(0, 2\sigma')$, $E\varepsilon_i = \sigma_i^*$</th>
<th>$Ee^{i\varepsilon\varepsilon'}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>All four paths the same</td>
<td>$N$</td>
<td>$\beta(\varepsilon) \beta(\varepsilon')$, $\varepsilon - N(0, 2\sigma')$, $\varepsilon - N(0, 2\sigma')$, $E\varepsilon_i = \sigma_i^*$</td>
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<td>$e^{&lt;}$</td>
</tr>
<tr>
<td>Two of the four paths are the same: $i = j$ xor</td>
<td>$2N(N-1)(N-2)$</td>
<td>$\beta(\varepsilon) \beta(\varepsilon')$, $\varepsilon - N(0, 2\sigma')$, $\varepsilon - N(0, 2\sigma')$, $E\varepsilon_i = 0$</td>
<td>$e^{&lt;}$</td>
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<td>... two of the four paths are the same: $i = i'$ xor</td>
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<td>$e^{&lt;}$</td>
</tr>
<tr>
<td>All four paths are distinct</td>
<td>$N(N-1)$</td>
<td>$\beta(\varepsilon) \beta(\varepsilon')$, $\varepsilon - N(0, 2\sigma')$, $\varepsilon - N(0, 2\sigma')$, $E\varepsilon_i = 0$</td>
<td>$e^{&lt;}$</td>
</tr>
</tbody>
</table>

Table 2. Expectations for un-calibrated cases.