Robust MVDR Beamforming Using Worst-Case Performance Optimization

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Introduction

Conditions required when designing an adaptive beamformer:

- if the signal component is present in the training data cell, the steering vector of the desired signal has to be known precisely
- large number of snapshots (training sample size)
- stationary training data set

Violation of any of these assumptions is known to lead to a significant performance breakdown of adaptive beamforming!
Introduction

Typical causes of steering vector mismatch:

- *look direction uncertainty and signal pointing errors*
- *poor array calibration*
- *unknown distortions of array geometry*
- *unknown sensor mutual coupling*
- *signal local scattering, source spreading, and fading effects*
- *near-far wavefront mismodeling*
- *unknown wavefront distortions and fluctuations*
Adaptive Beamforming: Background

The output of a narrowband beamformer:

\[ y(k) = w^H x(k) \]
\[ w = [w_1, \ldots, w_M]^T \]
\[ x(k) = [x_1(k), \ldots, x_M(k)]^T \]
Adaptive Beamforming: Background

Training snapshot vector:

\[ x(k) = s(k)a + i(k) + n(k) \]

Max signal-to-interference-plus-noise ratio (SINR) criterion:

\[ \max_w \text{SINR}, \quad \text{SINR} = \frac{\sigma_s^2 |w^H a|^2}{w^H R_{i+n} w} \]

Interference-plus-noise covariance matrix:

\[ R_{i+n} = \mathbb{E}\left\{ (i(k) + n(k)) (i(k) + n(k))^H \right\} \]
Adaptive Beamforming: Background

Equivalent MVDR beamforming problem:

$$\min_w w^H R_{i+n} w \quad \text{subject to} \quad w^H a = 1$$

Optimal weight vector:

$$w_{opt} = \frac{1}{a^H R_{i+n}^{-1} a} R_{i+n}^{-1} a$$

In practice, the interference-plus-noise covariance matrix $R_{i+n}$ is unavailable $\implies$ sample estimate $\hat{R}$ is used
Adaptive Beamforming: Background

Sample estimate of $R_{i+n}$:

$$\hat{R} = \frac{1}{N} \sum_{n=1}^{N} x(n)x^H(n) \quad \text{contains the signal component!}$$

where $N$ is the training sample size.

Practical formulation of the MVDR problem:

$$\min_{w} w^H \hat{R} w \quad \text{subject to} \quad w^H a = 1$$

Sample matrix inversion (SMI) algorithm [Reed, Mallett, Brennan]:

$$w_{SMI} = \hat{R}^{-1} a \quad \text{immaterial constant omitted}$$
Adaptive Beamforming: Background

Actual steering vector:

\[ \tilde{a} = a + \Delta \neq a \]

where \( \Delta \) is the unknown steering vector mismatch.

Why the robustness of the SMI beamformer is insufficient?

- **The main cause** is the presence of the signal component in the training data cell
- The signal is **misinterpreted** as an interference and is **suppressed** by means of adaptive nulling instead of being protected
- **Similar effect** when \( \Delta = 0 \) but the training sample size is small
Adaptive Beamforming: Background

Known approaches robust against arbitrary steering vector mismatch:

- Diagonally loaded SMI algorithm (LSMI)
- Eigenspace-based beamformer

LSMI algorithm [Cox et al], [Carlson]:

\[ w_{LSMI} = \hat{R}_{dl}^{-1} a, \quad \hat{R}_{dl} = \xi I + \hat{R} \]

Eigenspace-based beamformer [Yeh], [Feldman, Griffiths]:

\[ w_{eig} = \hat{R}^{-1} Pa \]

where \( P \) is the projector onto sample signal-plus-interference subspace
Robust MVDR Beamforming

Let the unknown steering vector mismatch $\Delta$ can be bounded by some known $\varepsilon > 0$:

$$\|\Delta\| \leq \varepsilon$$

Then, the actual signal spatial signature vector $\hat{a}$ belongs to the set

$$\mathcal{A}(\varepsilon) \triangleq \{ c \mid c = a + e, \|e\| \leq \varepsilon \}.$$ 

Vector $\hat{a}$ can be any vector in this set $\implies$ we impose a constraint

$$|w^H c| \geq 1 \quad \text{for all} \quad c \in \mathcal{A}(\varepsilon)$$
Robust MVDR Beamforming

Our formulation of the robust MVDR beamformer:

$$\min_{w} w^H \hat{R} w \quad \text{s. t.} \quad |w^H c| \geq 1 \quad \text{for all} \quad c \in A(\varepsilon)$$

- instead of fixed distortionless response towards the presumed steering vector $\mathbf{a}$, a soft distortionless response is maintained for a continuum of all possible steering vectors
- our constraints guarantee that the distortionless response will be maintained in the worst case
- infinite number of nonlinear and nonconvex constraints – how to solve?
Robust MVDR Beamforming

Equivalent problem:

$$\min_w w^H \hat{R} w \quad \text{subject to} \quad \min_{c \in \mathcal{A}(\varepsilon)} |w^H c| \geq 1$$

Result:

$$\min_{c \in \mathcal{A}(\varepsilon)} |w^H c| = |w^H a| - \varepsilon \|w\|$$

Using Result, we obtain an equivalent problem with a single non-linear and non-convex constraint:

$$\min_w w^H \hat{R} w \quad \text{subject to} \quad |w^H a| - \varepsilon \|w\| \geq 1 \quad (*)$$
Robust MVDR Beamforming

**Observation:** the cost function in (*) is unchanged when \( \mathbf{w} \) undergoes an arbitrary phase rotation \( \implies \) without any loss of generality, we can choose \( \mathbf{w} \) such that

\[
\begin{align*}
\text{Re} \left\{ \mathbf{w}^H \mathbf{a} \right\} & \geq 0, \\
\text{Im} \left\{ \mathbf{w}^H \mathbf{a} \right\} & = 0.
\end{align*}
\]

Employing these conditions as additional constraints, rewrite (*) as

\[
\begin{array}{ll}
\min_{\mathbf{w}} \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} & \text{ s. t. } \mathbf{w}^H \mathbf{a} \geq \varepsilon \|\mathbf{w}\| + 1, \text{ Im} \left\{ \mathbf{w}^H \mathbf{a} \right\} = 0 \\
& \text{ but this is the so-called convex second-order cone (SOC) program!}
\end{array}
\]
Robust MVDR Beamforming

**Comments to the solution:**

- **SOC problems** can be very efficiently solved using the well established *interior point method*
- Convex programming *software tools* are available in Matlab (e.g., *SeDuMi* [Sturm])
- **Computational complexity** of our robust beamformer is $O(M^{2.5})$ (compare it with $O(M^3)$ of the SMI beamformer!)
Simulations

**Common setup:**

- **ULA of** $M = 10$ omnidirectional sensors, *half-wavelength* spacing
- **100 independent runs** are used to obtain each simulated point
- **two interfering sources** with DOA’s $30^\circ$ and $50^\circ$ w.r.t. broadside
- interference-to-noise ratio **INR = 15 dB** in a single sensor
- the **SMI, LSMI, eigenspace-based**, and the **proposed** beamformers compared
- signal component is **always present** in the data snapshots
Simulations

Example 1:

• exactly known signal steering vector
• $\theta_s = 3^\circ$
• Fig. 1: compares the beamformers in terms of the mean output array SINR vs. $N$ for the fixed $\text{SNR} = -10$ dB
• Fig. 2: compares the beamformers in terms of the mean output array SINR vs. SNR for the fixed $N = 30$
Simulations

![Graph showing OUTPUT SINR (dB) vs. NUMBER OF SNAPSHOTs for different beamformers: OPTIMAL SINR, PROPOSED ROBUST BEAMFORMER, SMI BEAMFORMER, LSMI BEAMFORMER, and EIGENSPACE-BASED BEAMFORMER.]
Simulations

SNR (DB)

OUTPUT SINR (DB)

OPTIMAL SINR
PROPOSED ROBUST BEAMFORMER
SMI BEAMFORMER
LSMI BEAMFORMER
EIGENSPACE–BASED BEAMFORMER

SNR (DB)
Simulations

Example 2:

- signal \textit{look direction mismatch}
- both the \textit{presumed} and \textit{actual} signal steering vectors are \textit{plane waves} with the DOA’s 3° and 5°, respectively
- Fig. 3: compares the beamformers in terms of the \textit{mean output array SINR} vs. \( N \) for the fixed \( \text{SNR} = -10 \text{ dB} \)
- Fig. 4: compares the beamformers in terms of the \textit{mean output array SINR} vs. \( \text{SNR} \) for the fixed \( N = 30 \)
Simulations

OUTPUT SINR (DB)

NUMBER OF SNAPHOTS

OPTIMAL SINR
PROPOSED ROBUST BEAMFORMER
SMI BEAMFORMER
LSMI BEAMFORMER
EIGENSPACE–BASED BEAMFORMER
Simulations

![Graph showing output SINR vs. SNR for different beamformers.]

- **Green line**: OPTIMAL SINR
- **Red line**: PROPOSED ROBUST BEAMFORMER
- **Blue diamonds**: SMI BEAMFORMER
- **Turquoise triangles**: LSMI BEAMFORMER
- **Pink diamonds**: EIGENSPACE–BASED BEAMFORMER

SNR (DB) vs. OUTPUT SINR (DB) graph. The graph demonstrates the performance of different beamforming techniques under varying Signal-to-Noise Ratio (SNR) conditions.
Simulations

Example 3:

- signal steering vector mismatch due to \textit{coherent local scattering}
- the \textit{presumed} signal steering vector is a \textit{plane wave} from $3^\circ$
- the \textit{actual} signal steering vector is formed by \textit{five signal paths} as
  \[ \tilde{\mathbf{a}} = \mathbf{a} + \sum_{i=1}^{4} e^{j\psi_i} \mathbf{b}(\theta_i) \]
- $\mathbf{a}$ is the \textit{direct path} and $\mathbf{b}(\theta_i)$ are \textit{scattered paths};
- $\theta_i$ from uniform random generator with \textit{mean} $= 3^\circ$ and \textit{std} $= 2^\circ$
- Fig. 5: compares the beamformers in terms of the \textit{mean output array SINR} vs. $N$ for the fixed $\text{SNR} = -10$ dB
- Fig. 6: compares the beamformers in terms of the \textit{mean output array SINR} vs. $\text{SNR}$ for the fixed $N = 30$
Simulations

![Graph showing output SINR vs SNR for various beamformers: Optimal SINR, Proposed Robust Beamformer, SMI Beamformer, LSMI Beamformer, and Eigenspace-Based Beamformer.](image-url)
Simulations

Example 4:

- *near-far mismodeling* of the signal wavefront
- the *presumed* signal steering vector is a *plane wave* from 0°
- the *actual* signal steering vector is formed by the source located in the *near field* of the antenna at a distance \( \frac{D^2}{\lambda} = (M - 1)^2 \frac{\lambda}{4} \) from the *geometrical center* of the array
- Fig. 7: compares the beamformers in terms of the *mean output array SINR* vs. \( N \) for the fixed \( \text{SNR} = -10 \text{ dB} \)
- Fig. 8: compares the beamformers in terms of the *mean output array SINR* vs. \( \text{SNR} \) for the fixed \( N = 30 \)
Simulations

![Graph showing the comparison of different beamforming techniques.](graph.png)
Simulations

![Graph showing output SINR versus SNR for different beamformers: Optimal SINR, Proposed Robust Beamformer, SMI Beamformer, LSMI Beamformer, and EIGENSPACE-Based Beamformer.](image-url)
Simulations

Example 5:

- signal steering vector mismatch due to a distortion of the signal wavefront which is related to wave propagation effects
- the presumed signal steering vector is a plane wave from $3^\circ$
- the actual signal steering vector involves independent-increment phase distortions which are independently drawn in each run from a Gaussian generator with the variance 0.04
- Fig. 9: compares the beamformers in terms of the mean output array SINR vs. $N$ for the fixed $\text{SNR} = -10$ dB
- Fig. 10: compares the beamformers in terms of the mean output array SINR vs. SNR for the fixed $N = 30$
Simulations

![Graph showing output SINR vs. number of snapshots for different beamformers]

- **Optimal SINR**
- **Proposed Robust Beamformer**
- **SMI Beamformer**
- **LSMI Beamformer**
- **Eigenspace-Based Beamformer**

**X-axis:** Number of Snapshots

**Y-axis:** Output SINR (dB)
Simulations

SNR (DB)

OUTPUT SINR (DB)

OPTIMAL SINR
PROPOSED ROBUST BEAMFORMER
SMI BEAMFORMER
LSMI BEAMFORMER
EIGENSPACE–BASED BEAMFORMER

SNR (DB)
Conclusions

- *new robust approach* to adaptive beamforming has been proposed for the case of *arbitrary steering vector mismatches*
- our technique is based on the *worst-case performance optimization* using *convex SOC programming*
- significant *robustness improvements* with respect to existing methods are achieved
- the proposed beamformer provides *substantial performance improvements* even in the case of *exactly known* steering vector but *small* training sample size
- the computational cost is *comparable* to that of the *conventional adaptive beamforming methods* (such as SMI or LSMI)