Direction Finding for Tracking Radar Using a Simplified, Subspace Based Approach

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Using the spatial information available from an array of sensors to overcome target ‘glint’
Multiple, coherent signals

- Multiple coherent signals can interfere at the receiver
- Create non-plane waves at the antenna
- Change the response of conventional beams
- Can cause monopulse direction finding to fail
Interference effects at the array

- Spatial variations in amplitude and phase gradient

Many point scatterers on an aircraft

Field magnitude in the region of the array

Regions of abnormally high phase rate

Regions of constructive and destructive interference

Instantaneous phase gradient in the region of the array
Conventional beamforming

- Project a manifold vector onto a signal vector
Multi-dimensional beamforming

- Project a signal \textit{vector} onto a manifold \textit{subspace}
- \textit{Maximum likelihood} approach

\begin{itemize}
  \item Manifold \textit{subspace} is a plane or volume
  \item Signal vector
  \item Dimension 1
  \item Dimension 2
  \item Dimension 3
  \item Maximum likelihood scan
  \item Scan in azimuth
  \item Scan in elevation
  \item No error
  \item Magnitude of ‘p’
\end{itemize}
Multi-dimensional beamforming

- Project a manifold vector onto a measured signal subspace
- MUSIC approach, using spatial smoothing

![Diagram showing projection of manifold vector on orthogonal and signal subspaces, with dimensions 1, 2, and 3. A MUSIC scan is also depicted with low error and magnitude of 'p'.]
Signal model

- Single scatterer in far field gives rise to plane waves
- Multiple scatterers create composite waves with 3 significant planar components

![Diagram showing signal model with scatterers at different angles, planar wavefronts, and elevation and azimuth differences.](Image)
Results

- Combined signal from many scatterers can be approximately modelled by a simple linear combination of three waveforms.
- Multi-dimensional projection approaches such as MUSIC or Maximum Likelihood can reduce or eliminate glint errors subject to low noise levels.

![Graphs showing RMS deviation of position estimates and mean error in position estimates.](image-url)
Bounds on performance

- Can use the *uniform* Cramer-Rao lower bound (CRB) \(^5\) to show limits on theoretical performance for a given scatterer arrangement, signal to noise ratio, and receiver array.

- Assume that *the scatterers are all close together* and use an invertible transform to obtain the mean scatterer position.

- Generate the uniform CRB on estimates of this mean position.

- Use targets containing single and multiple scatterers with constructive and destructive interference at the receiver.
Estimation of a single parameter, \( u \) from the signal received on a linear array of sensors.
sources in phase

Estimation of a single parameter, the mean of $u$ from the signal received on a linear array of sensors
Uniform CR bound - 4 30dB sources

Estimation of a single parameter, the mean of $\mu$ from the signal received on a linear array of sensors
Conclusions

- Monopulse estimation does not always achieve performance equal (or near) to the uniform CRB and better estimators can exist.

- The close proximity of scatterers within a target allows the received signal to be modelled by a subspace of low dimension.

- Multi-dimensional beamforming approaches such as MUSIC and maximum likelihood can use this simplified signal model to offer improvements in bias (compared with monopulse) without significant increase in variance.
Definition

- Let $F_z(z; \theta)$ be the natural logarithm of the probability density of vector random variable $z$, with $\theta$ a vector of nonrandom parameters.
- Then $F_z(z; \theta)$ is said to be the log-likelihood function for $\theta$ [1].
- Let $\theta$ be partitioned as $[\phi^T \ \eta^T]^T$, where $\phi$ is a vector of parameters of interest and $\eta$ is a vector of unwanted parameters.
- Then $F_z^p(z; \phi) \triangleq \sup_\eta F_z(z; \theta)$ is said to be the profile log-likelihood for $\phi$.

Direction-of-arrival Estimation

### Signal model

- Let there be $n$ sources incident on a planar array, with steering vectors $\mathbf{a}(u_i, v_i), i = 1, n$ and complex amplitudes $\alpha_i, i = 1, n$

- Let a single vector observation at the array be denoted $\mathbf{z} = \mathbf{A}\alpha + \varepsilon$ where

\[
\mathbf{A} \triangleq \begin{bmatrix}
\mathbf{a}(u_1, v_1) & \cdots & \mathbf{a}(u_n, v_n)
\end{bmatrix}, \quad \alpha \triangleq \begin{bmatrix}
\alpha_1 & \cdots & \alpha_n
\end{bmatrix}^T
\]

and $\varepsilon$ is a drawn from a zero mean multivariate normal distribution, with covariance $\sigma^2 \mathbf{I}$.
Lemma 1

- The profile log-likelihood for

\[ \phi \triangleq \begin{bmatrix} u_1 & \cdots & u_n & \nu_1 & \cdots & \nu_n \end{bmatrix}^T \]

is

\[ F_z^p(z; \phi) = \text{const} - \frac{1}{\sigma^2} (z^H (I - AA^+) z) \]

Proof

- The observation \( z \) is parameterised by the vector

\[ \theta \triangleq \begin{bmatrix} \phi^T & \alpha^T \end{bmatrix}^T \]

with log-likelihood \([2]\)

\[ F_z(z; \theta) = \text{const} - \frac{1}{\sigma^2} (z - A\alpha)^H (z - A\alpha) \quad (1) \]

- By definition the minimum norm solution of \( z - A\alpha \)

is \( \alpha = A^+ z \), where \( A^+ \) is the pseudo-inverse of \( A \).

– It follows directly that (1) is maximised by \( \alpha = A^+z \) for all \( A \), and substituting into (1) we get

\[
F^p_z(z; \phi) = \text{const} - \frac{1}{\sigma^2}(z - AA^+z)^H(z - AA^+z)
\]

\[
= \text{const} - \frac{1}{\sigma^2}(z^H(I - AA^+)z)
\]

**Comments**

– When \( A \) has full column rank \( A^+ = (A^H A)^{-1} A^H \), and the profile log-likelihood takes the more familiar form

\[
F^p_z(z; \phi) = \text{const} - \frac{1}{\sigma^2}(z^H(I - A(A^H A)^{-1} A^H)z)
\]
Lemma 2a

\[
A^H \frac{\partial (AA^+)}{\partial \phi_i} = \frac{\partial A^H}{\partial \phi_i} (I - AA^+)
\]

Proof

- From the Moore-Penrose conditions \cite{3} we have
  \[
  AA^+A = A \quad \quad A^H A^+A = A^H
  \]
- Differentiating both sides wrt to \( \phi_i \) gives
  \[
  \frac{\partial A^H}{\partial \phi_i} A^+A + A^H \frac{\partial (A^+A)}{\partial \phi_i} = \frac{\partial A^H}{\partial \phi_i}
  \]
  and the result follows directly.

Lemma 2b

\[
\frac{\partial^2 (AA^+)}{\partial \phi_i \partial \phi_k} A = - \frac{\partial (AA^+)}{\partial \phi_i} \frac{\partial A}{\partial \phi_k} - \frac{\partial (AA^+)}{\partial \phi_k} \frac{\partial A}{\partial \phi_i} + (I - AA^+) \frac{\partial^2 A}{\partial \phi_i \partial \phi_k}
\]

Proof

- Starting with \(AA^+ A = A\), differentiate both sides by \(\phi_i\) and \(\phi_k\), and the result follows directly.
Lemma 3

- If $\varepsilon$ is drawn from a $m$-variate, zero-mean, Normal distribution with covariance $\sigma^2 \mathbf{I}$, then $\varepsilon^H \mathbf{A} \mathbf{A}^+ \varepsilon$ is Chi-squared distributed with $2r$ degrees of freedom, where

$$r = rank(\mathbf{A} \mathbf{A}^+) = rank(\mathbf{A})$$

Proof

- The result follows directly from the fact that $\mathbf{A} \mathbf{A}^+$ is idempotent (i.e. $\mathbf{A} \mathbf{A}^+ \mathbf{A} \mathbf{A}^+ = \mathbf{A} \mathbf{A}^+$) via application of Muirhead, theorem 1.4.2 [4].

Theorem 1

The Fisher information matrix, \( J_p(\phi) \), corresponding to the profile log-likelihood \( F_z^p(z; \phi) \) can be written as

\[
J_p(\phi) = \frac{2}{\sigma^2} \Re \begin{bmatrix}
X^H D_u^H (I - AA^+) D_u X & X^H D_u^H (I - AA^+) D_v X \\
X^H D_v^H (I - AA^+) D_u X & X^H D_v^H (I - AA^+) D_v X
\end{bmatrix}
\]

where \( X \triangleq \text{diag}(\alpha) \), \( D_u \triangleq \begin{bmatrix}
\frac{\partial a(u_1, v_1)}{\partial u_1} & \ldots & \frac{\partial a(u_n, v_n)}{\partial u_n}
\end{bmatrix} \),

and \( D_v \triangleq \begin{bmatrix}
\frac{\partial a(u_1, v_1)}{\partial v_1} & \ldots & \frac{\partial a(u_n, v_n)}{\partial v_n}
\end{bmatrix} \).
Proof

\[ [J^p(\phi)]_{ik} \triangleq -E \left\langle \frac{\partial^2 F_z^p(z;\phi)}{\partial \phi_i \partial \phi_k} \right\rangle \]

- According to lemma 1

\[ [J^p(\phi)]_{ik} \triangleq -\frac{1}{\sigma^2} \left\{ \alpha^H A^H \frac{\partial^2 (AA^+)}{\partial \phi_i \partial \phi_k} A \alpha \right\} \]

\[ + E \left\langle \varepsilon^H \right\rangle \frac{\partial^2 (AA^+)}{\partial \phi_i \partial \phi_k} A \alpha \]

\[ + \alpha^H A^H \frac{\partial^2 (AA^+)}{\partial \phi_i \partial \phi_k} E \left\langle \varepsilon \right\rangle \]

\[ + E \left\langle \varepsilon^H \frac{\partial^2 (AA^+)}{\partial \phi_i \partial \phi_k} \varepsilon \right\rangle \]

(2)
According to lemma 2, the first term in (2) can be written as

\[ \alpha^H \frac{\partial A^H}{\partial \phi_i} (I - AA^+) \frac{\partial A}{\partial \phi_k} \alpha \]

\[ + \alpha^H \frac{\partial A^H}{\partial \phi_k} (I - AA^+) \frac{\partial A}{\partial \phi_i} \alpha \]

\[ + \alpha^H A^H (I - AA^+) \frac{\partial^2 A}{\partial \phi_i \partial \phi_k} \alpha \]

Since \( A^H AA^+ = A^H \), the last term is zero and the first two terms have the more compact form

\[ 2 \text{Re} \left[ X^H D_u^H (I - AA^+) D_u X \quad X^H D_u^H (I - AA^+) D_v X \right] \]

\[ \left[ X^H D_v^H (I - AA^+) D_u X \quad X^H D_v^H (I - AA^+) D_v X \right]_{ik} \]
The second and third terms in (2) are zero because $\varepsilon$ has zero mean.

The last term in (3) can be written as

$$E\left(\varepsilon^H \frac{\partial^2 (AA^+)}{\partial \phi_i \partial \phi_k} \varepsilon\right) = \frac{\partial^2}{\partial \phi_i \partial \phi_k} E\left(\varepsilon^H A A^+ \varepsilon\right)$$

According to lemma 3 $\varepsilon^H A A^+ \varepsilon$ is distributed with mean and variance which are independent of $A$ and consequently of $\phi$.

The last term in (2) is therefore equal to zero, and the theorem is proved.
Comments

- For linear arrays $A$ has no dependence on $\nu$.
- When $A$ has full column rank $J^p(\phi)$ takes the form

$$\frac{2}{\sigma^2} \text{Re}\{X^H D_u^H (I - A \begin{pmatrix} A^H A \end{pmatrix}^H A^H) D_u X\}$$

which is familiar as the inverse of the bound derived in [2].
Cramer-Rao Bounds

- We can use the Fisher information matrix to bound the variance of an estimator \( \hat{t}(z) \) of the scalar function \( t(\phi) \).
- The general form of the bound is as follows \(^5\)

\[
\sigma_{\phi}^2 \geq (\nabla_\phi t + \nabla_\phi b)^T J^{p-1} (\nabla_\phi t + \nabla_\phi b)
\]

where \( b \triangleq E\{\hat{t}(z) - t(\phi)\} \)

Estimating the mean direction-of-arrival

For a group of closely spaced point sources it may be sufficient to estimate the mean of each coordinate.

Let \( t(\phi) = \frac{1}{n} \sum_i u_i \)

We can apply an invertible transformation of the parameter space, \( \tilde{\phi} = T\phi \).

Then \( \nabla_{\tilde{\phi}} \tilde{t} = T^{-T} \nabla_{\phi} t \) and \( \tilde{J}^{p-1} = T J^{p-1} T^T \).

To explore the bias-variance tradeoffs involved in estimating the mean doa, choose \( T \), so that

\[
\nabla_{\tilde{\phi}} \tilde{t} = [1 \ 0 \ \cdots \ 0]^T
\]
Uniform Bound

Bias Gradient norm

- Let $\delta^2 \triangleq \nabla_{\tilde{\phi}} \tilde{b}^T C \nabla_{\tilde{\phi}} \tilde{b}$, where $C$ is a diagonal matrix;

$$C = \begin{bmatrix}
\frac{b^2}{2} & 0 & \cdots & 0 \\
0 & \frac{\Delta u^2}{2} & \cdots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \frac{\Delta u^2}{2}
\end{bmatrix}$$

with $b$ equal to the beamwidth of the array, and $\Delta u$ equal to the dimension of the target.

- Expressions in $[5]$ allow us to compute the uniform CR bound.